Estimation of Small Perturbations in an Inertial Sensor

Pedro E. Zadunaisky* and Ricardo S. Sanchez Peña†
IIAE—Centro Espacial San Miguel, San Miguel, Argentina

An inertial sensor that can be simulated mathematically by a second-order ordinary differential equation is considered. Such an equation involves an unknown perturbing function that represents the effects of mass unbalance and anisoelasticity. Assuming that the solution of the differential equation and its first derivative are known measurable outputs from the sensor, a stepwise determination of the perturbation is made through a deterministic method, and upper bounds for the errors of the determination are established. The method has been tested in numerous computer simulations, and several successful examples are given.

I. Introduction

LET us consider a dynamic system that can be represented by a differential equation of the form

$$\ddot{y}=f(y,\dot{y},t)+P(t) \tag{1}$$

where y and $\dot{y}$ are measurable outputs, $f(y,\dot{y},t)$ is a known function depending on the mathematical laws governing the system, and $P(t)$ is an unknown small perturbation to be determined on the basis of a set of measurements of the quantities y and $\dot{y}$; in fact, it is assumed that $y(t_n)$ and $\dot{y}(t_n)$ are measured on a discrete set of points $t_n (n=1,2,\ldots)$. The classical solution of such a problem consists of the parameter identification of a model of the perturbation $P(t)$ by means of an overdetermined system of equations of condition. However, the first author has been able to develop a direct method that allows the estimation of P at points t_n, avoiding the necessity of establishing any a priori model for $P(t)$.[3,4]

In the following sections, we give the mathematical backgrounds and formulas for the application of the method. We also establish bounds for errors in the estimation of $P(t)$ resulting from our discretization schemes and from measurement errors in $y(t)$ and $\dot{y}(t)$.

In order to test the method, we simulated an inertial sensor subject to a given perturbation $P(t)$ and with certain assumed initial conditions. On this basis, we created "measurements" of $y(t_k)$ and $\dot{y}(t_k)$, including Gaussian random errors, with which we calculated by our method approximations $\tilde{P}(t_k)$ that were compared to the actual values $P(t_k)$. Numerous tests were performed, and we give some of the results obtained.

The last section is devoted to an analysis of these results and some final conclusions.

II. Mathematical Background

Let us consider a discrete set of equidistant points t_n $(n=1,2,\ldots)$, such that $t_{n+1}=t_n+h$, where h is a constant.

This condition of equidistance is not strictly necessary, and we adopt it simply to avoid some formal complications. Furthermore, let us assume that the solution of Eq. (1) may be represented by a convergent Taylor expansion, with the remainder expressed in integral form, such that

$$y(t_k)=y(t_j)+h\dot{y}(t_j)+\ldots+\frac{h^p}{p!}y^{(p)}(t_j)+\frac{1}{p!}\int_{t_j}^{t_k}y^{(p+1)}(u)(t_k-u)^p\mathrm{d}u \tag{2}$$

where $|t_k-t_j|=h$.

For $p=1$, we have

$$y(t_k)=y(t_j)+h\dot{y}(t_j)+\int_{t_j}^{t_k}\ddot{y}(u)(t_k-u)\mathrm{d}u \tag{3}$$

and, by virtue of Eq. (1),

$$y(t_k)=y(t_j)+h\dot{y}(t_j)+\int_{t_j}^{t_k}[f(y,\dot{y},u)+P(u)](t_k-u)\mathrm{d}u \tag{4}$$

Now, let us consider a "reference" problem,

$$\ddot{y}^j=f(y^j,\dot{y}^j,t)$$

obtained from Eq. (1) by dropping the unknown perturbation $P(t)$ and assuming the osculating initial conditions

$$y^j(t_j)=y(t_j) \qquad \dot{y}^j(t_j)=\dot{y}(t_j) \tag{5}$$

These quantities will later be assumed as known from measurements and affected in consequence by random errors (see Sec. III). Then, we have

$$y^j(t_k)=y^j(t_j)+h\dot{y}^j(t_j)+\int_{t_j}^{t_k}f(y^j,\dot{y}^j,u)(t_k-u)\mathrm{d}u \tag{6}$$

comparing with Eq. (4) and, by virtue of Eq. (5),

$$y(t_k)-y^j(t_k)=\int_{t_j}^{t_k}[f(y,\dot{y},u)-f(y^j,\dot{y}^j,u)+P(u)](t_k-u)\mathrm{d}u \tag{7}$$

From now on, we shall put

$$R_k^j=y(t_k)-y^j(t_k) \tag{8}$$

which is the difference, or "residual," between the value of the actual solution $y(t_k)$ of Eq. (1) at point t_k and the corresponding value $y^j(t_k)$ of the reference solution fulfilling the osculating conditions (5) at point t_j.

Received Nov. 7, 1986; revision received April 23, 1987.

*Principal Researcher.
†Associate Researcher; currently, Graduate Student, California Institute of Technology. Student Member AIAA.

Furthermore, let us write for the expression in brackets of Eq. (7)

$$\phi^j(u) = f(y,\dot{y},u) - f(y^j,\dot{y}^j,u) + P(u) \tag{9}$$

and Eq. (7) takes the form

$$R_k^j = \int_{t_j}^{t_k} \phi^j(u)(t_k - u)\mathrm{d}u \tag{10}$$

Let us consider three successive points t_{n-1}, t_n, t_{n+1} and define a quadratic interpolating function

$$z(u) = a + b(t_k - u) + c(t_k - u)^2 \tag{11}$$

such that $z(u) = \phi(u)$ at the three points. The coefficients a, b, and c depend on the reference point t_k. For instance, if we take $k = n-1$, we have

$$a = \phi^j(t_{n-1})$$

$$b = \frac{1}{2h}[3\phi^j(t_{n-1}) - 4\phi^j(t_n) + \phi^j(t_{n+1})]$$

$$c = \frac{1}{2h^2}[\phi^j(t_{n-1}) - 2\phi^j(t_n) + \phi^j(t_{n+1})]$$

Replacing $z(u)$ by $\phi(u)$ in Eq. (10) and integrating, we obtain, for $j = n$,

$$R_{n-1}^n = h^2\left[\frac{1}{8}\phi^n(t_{n-1}) + \frac{5}{12}\phi^n(t_n) - \frac{1}{24}\phi^n(t_{n+1})\right] + \delta I \tag{12}$$

where δI is the error introduced by replacing $z(u)$ by $\phi^j(u)$. Now, let us put

$$\Delta f_k^j = f[y(t_k),\dot{y}(t_k),t_k] - f[y^j(t_k),\dot{y}^j(t_k),t_k] \tag{13}$$

and

$$\tilde{R}_k^j = R_k^j/h^2 \tag{14}$$

Obviously, $\Delta f_k^j = 0$ for $j = k$, and Eq. (12), by virtue of Eq. (9), reduces to the form

$$\tilde{R}_{n-1}^n + \frac{1}{24}\Delta f_{n=1}^n - \frac{1}{8}\Delta f_{n+1}^n - \frac{\delta I}{h^2} = \frac{1}{8}P(t_{n-1}) + \frac{5}{12}P(t_n) - \frac{1}{24}P(t_{n+1}) \tag{15}$$

The same reasoning can be applied by combining the three points in several different pairs; by taking, for instance,

$$\begin{array}{ll} k = n-1 & j = n \\ k = n & j = n-1 \\ k = n & j = n+1 \\ k = n+1 & j = n \end{array} \tag{16}$$

we obtain a system of four linear equations for $P(t_{n-1})$, $P(t_n)$, and $P(t_{n+1})$ as follows:

$$\begin{bmatrix} \frac{1}{8} & \frac{5}{12} & \frac{-1}{24} \\ \frac{7}{24} & \frac{1}{4} & \frac{-1}{24} \\ \frac{-1}{24} & \frac{1}{4} & \frac{7}{24} \\ \frac{-1}{24} & \frac{5}{12} & \frac{1}{8} \end{bmatrix} \begin{bmatrix} P(t_{n-1}) \\ P(t_n) \\ P(t_{n+1}) \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{R}_{n-1} - \frac{1}{8}\Delta f_{n-1}^n + \frac{1}{24}\Delta f_{n+l}^n - \delta I/h^2 \\ \tilde{R}_n^{n-1} - \frac{7}{24}\Delta f_n^{n-l} + \frac{1}{24}\Delta f_{n+1}^{n-1} - \delta I/h^2 \\ \tilde{R}_n^{n+1} + \frac{1}{24}\Delta f_{n-1}^{n+1} - \frac{7}{24}\Delta f_n^{n+1} - \delta I/h^2 \\ \tilde{R}_{n+1}^n + \frac{1}{24}\Delta f_{n-1}^n - \frac{1}{8}\Delta f_{n+1}^n - \delta I/h^2 \end{bmatrix} \tag{17}$$

where the first equation is precisely Eq. (15).

If we write this system in the form

$$MP = \tilde{R} \tag{18}$$

then $M \epsilon R^{4\times 3}$ is a rectangular matrix, $P = [P(t_{n-1}), P(t_n), P(t_{n+1})]^T$ and $\tilde{R}$ is the vector at the right-hand member of Eq. (17). This linear system is overdetermined, and we may obtain the generalized inverse of M,

$$M^+ = (M^TM)^{-1}M^T \tag{19}$$

which, in this case, is exactly

$$M^+ = \begin{bmatrix} -0.9 & 3.7 & 1.3 & -2.1 \\ 1.5 & -0.5 & -0.5 & 1.5 \\ -2.1 & 1.3 & 3.7 & -0.9 \end{bmatrix} \tag{20}$$

Then, we have

$$P = M^+\tilde{R} \tag{21}$$

III. Error Bounds

In Eq. (21), the generalized inverse M^+ has no errors and, if $\tilde{R}$ is affected by some errors $\delta\tilde{R}$, which we are going to analyze, then we have, for the errors in the unknowns,

$$\delta P = M^+\delta\tilde{R} \tag{22}$$

Inherent Errors

This kind of error stems form the approximations introduced by the method. The integral of Eq. (10) may be written, by the generalized mean value theorem for integrals, in the form

$$I_n^{n+1} = (t_{n+1} - \xi)\int_{t_n}^{t_{n+1}} \phi^n(u)\mathrm{d}u \tag{23}$$

with $\xi \epsilon (t_n, t_{n+1})$, so that $\max(t_{n+1} - \xi) \leq h$. The replacement of the integrand by the quadratic interpolant (11) is equivalent to Simpson's rule for integrals, and it is well known that the approximation error has the form $[-h^5\phi^{n(iv)}(h)/90]$ with $h\epsilon(t_n,t_{n+1})$; the total error in I_n^{n+1} is then $\|\delta I\| = \|h^6\phi^{n(iv)}(h)/90\|$ and, in view of Eq. (22), the inherent error is

$$\|\epsilon P \text{inherent}\| \leq \|M + \| \left(\frac{\delta \tilde{I}}{h^2}\right)$$

where $\delta\tilde{I}/h^2$ is a vector of four elements of the form

$$|h^4\phi_i^{h(iv)}(h)/90| \qquad (i=1,2,3,4) \tag{24}$$

Measurement Errors

In expression (8) for R_k^j, the solution $y(t)$ of Eq. (1) may be given exactly in the case of a control problem. Otherwise, $y(t_n)$ and $\dot{y}(t)$ may be given as quantities measured in a set of points t_k, so that

$$y(t_k) = \tilde{y}(t_k) - \epsilon_k$$

$$\dot{y}(t_k) = \tilde{\dot{y}}(t_k) - \dot{\epsilon}_k$$

$$k = 1,2,\ldots \tag{25}$$

where ϵ_k and $\dot{\epsilon}_k$ are measurement errors. These errors may affect the right-hand member of Eq. (17) in two ways.

In fact, if in Eq. (6), we replace $y(t_k)$, $y^j(t_j)$, and $\dot{y}^j(t_j)$ by the measured quantities $\tilde{y}(t_k)$, $\tilde{y}^j(t_j)$, and $\tilde{\dot{y}}^j(t_j)$, respectively, we introduce in R_k^j defined by Eq. (14) an error of the form

$$\epsilon_I = (\epsilon_k + \epsilon_j + h\dot{\epsilon}_j)/h^2 \tag{26}$$

Similarly, if instead of Eq. (13), we put

$$\Delta f_k^j \cong f[\tilde{y}(t_k), \tilde{\dot{y}}(t_k), t_k] - f[\tilde{y}^j(t_k), \tilde{\dot{y}}^j(t_k), t_k]$$

we introduce an error of the form

$$\epsilon_{II} = \frac{\delta f}{\delta y}(\epsilon_k + \epsilon_j + h\dot{\epsilon}_j) - \frac{\delta f}{\delta y}(\dot{\epsilon}_k + \dot{\epsilon}_k) \tag{27}$$

where, by virtue of Eq. (1),

$$\dot{\epsilon}_h = \dot{\epsilon}_j + h[f(y^j(t_j), \dot{y}^j(t_j), t_j) + P(t_j)] \tag{28}$$

Summarizing, we may say that the inherent error is proportional to h^4; the measurement errors ϵ_I are proportional in part to $1/h^2$, while ϵ_{II} is proportional to h. This indicates that when possible, in order to maintain the effect of these errors within acceptable limits, one should choose for the interval h a value of compromise. Equations (24), (26), (27), and (28) may help to make a proper analysis in any particular problem or situation.

IV. Simulation of the Inertial Sensor

To simulate the inertial sensor, we have adopted the following differential equation of the second order:

$$\ddot{y} + 2\omega_n \xi \dot{y} + \omega_n^2 y = K(\omega_x + T_{mt}/H + E_x a_x + E_z a_z + E_{xz} a_x a_z) \tag{29}$$

where

y = output from the signal generator

H = angular moment of the wheel (gyroscope) or mass unbalance (accelerometer)

ω_n = natural frequency of the sensor

$\omega_x, \omega_y, \omega_z$ = components of the inertial angular rate

ξ = damping factor

E_x, E_z = mass unbalance error factors along the x and z axis, respectively

E_{xz} = anisoelasticity error factor

K = scale factor

T_{mt} = torque applied by the torque motor

a_x, a_z = accelerations along the x and z axis, respectively

In this model, we have neglected errors stemming from the following sources: 1) cross coupling, 2) angular acceleration int he output axis appearing in gyroscopes, and 3) constant and random erros. We proceeded in this way in order to generate a model representing equally well both types of sensors. Anyway, the constant errors may be taken as errors in the term T_{mt}, which does not complicate the equations. Furthermore, the type of input we shall introduce will cancel the remaining errors. Therefore, such input will basically enhance the errors depending on the acceleration on both axes and that of anisoelasticity depending on the product of acceleration on those axes.

In the present simulation, we adopted:

$$\omega_x = 0, \quad \omega_y = \text{const}, \quad T_{mt} = H \text{ desired } (\omega_x)$$

$$a_x = g\sin(\omega_y t), \quad a_z = g\cos(\omega_y t) \tag{30}$$

Note that although ω_x is zero, we can have a desired (ω_x) different from zero, the reason being that we are actually simulating an inertial angular velocity ω_x through the torque generator T_{mt}. With reference to Eq. (1), we have

$$f(y,t) = -2\omega_n \xi \dot{y} - \omega_n^2 y + K(T_{mt}/H) \tag{31}$$

as the known part of the right-hand member and

$$P(t) = E_x g\sin(\omega_y t) + E_z g\cos(\omega_y t) + (E_{xz}/2)g^2\sin(2\omega_y t) \tag{32}$$

the unknown perturbation to be determined by our method. With these assumptions and with the initial conditions

$$y(t_0) = y_0$$

$$\dot{y}(t_0) = \dot{y}_0 \tag{33}$$

where y_0 and $\dot{y}_0$ are measured quantities, the solution of Eq. (1) may be expressed in closed form as follows:

$$\begin{aligned}
y(t) = {} & \frac{K\omega_x}{\omega_n^2} + \frac{e^{-\xi\omega_n t}\sin(\omega_n\sqrt{1-\xi^2 t})}{\omega_n\sqrt{1-\xi^2}} \Bigg\{ \dot{y}_0 + y_0\omega_n\xi \\
& - \frac{K\omega_x\xi}{\omega_n} + \frac{gE_x\omega_y[(2\xi^2-1)\omega_n^2+\omega_y^2] - gE_z\omega_n\xi(\omega_n^2+\omega_y^2)}{(\omega_n^2-\omega_y^2)^2+4\xi^2\omega_n^2\omega_y^2} \\
& + \frac{2g^2E_{xz}\omega_y[(2\xi^2-1)\omega_n^2+4\omega_y^2]}{(\omega_n^2-4\omega_y^2)^2+16\xi^2\omega_n^2\omega_y^2} \Bigg\} \\
& + e^{-\xi\omega_n t}\cos(\omega_n\sqrt{1-\xi^2 t}) \Bigg\{ y_0 - \frac{K\omega_x}{\omega_n^2} \\
& + \frac{2gE_x\omega_n\omega_y\xi - gE_z(\omega_n^2-\omega_y^2)}{(\omega_n^2-\omega_y^2)^2+4\xi^2\omega_n^2\omega_y^2} + \frac{4g^2E_{xz}\omega_n\omega_y\xi}{(\omega_n^2-4\omega_y^2)^2+16\xi^2\omega_n^2\omega_y^2} \Bigg\} \\
& + \sin(\omega_y t)\left[\frac{(\omega_n^2-\omega_y^2)gE_x + 2gE_z\xi\omega_n\omega_y}{(\omega_n^2-\omega_y^2)^2+4\xi^2\omega_n^2\omega_y^2}\right] \\
& + \cos(\omega_y t)\left[\frac{(\omega_n^2-\omega_y^2)gE_z - 2gE_x\xi\omega_n\omega_y}{(\omega_n^2-\omega_y^2)^2+4\xi^2\omega_n^2\omega_y^2}\right] \\
& + \frac{\sin(2\omega_y t)}{2}\left[\frac{E_{xz}g^2(\omega_n^2-4\omega_y^2)}{(\omega_n^2-4\omega_y^2)^2+16\xi^2\omega_n^2\omega_y^2}\right] \\
& - \frac{\cos(2\omega_y t)}{2}\left[\frac{E_{xz}4g^2\omega_n\omega_y\xi}{(\omega_n^2-4\omega_y^2)^2+16\xi^2\omega_n^2\omega_y^2}\right]
\end{aligned} \tag{34}$$

From this formula, it is easy to derive $\dot{y}(t)$, also in closed form. Of course, in the application of the method, the subindex "0" must be substituted by the set of subindices "j" defined by Eq. (16).

For the reference equation

$$\ddot{y}^r = -2\omega_n \xi \dot{y}^r - \omega_n^2 y^r + K \frac{T_{mt}}{H} \tag{35}$$

with the initial conditions

$$y^r(t_0) = y(t_0) = y_0$$
$$\dot{y}^r(t_0) = \dot{y}(t_0) = \dot{y}_0 \tag{36}$$

the solution is obtained simply by dropping in Eq. (34) all the terms containing as factors E_x, E_z, or E_{xz}.

Thus, we can obtain for any instant the corresponding values $y(t)$ and $\dot{y}(t)$ and then simulate measurements $\tilde{y}(t)$ and $\tilde{\dot{y}}(t)$ by

$$\tilde{y}(t) = y(t) + \epsilon_t \qquad \tilde{\dot{y}}(t) = \dot{y}(t) + \dot{\epsilon}_t \tag{37}$$

where ϵ_t and $\dot{\epsilon}_t$ are random numbers with a Gaussian distribution with zero mean and a variance σ.

V. Numerical Schemes

In all our numerical experiments, which we shall describe later, we applied Eq. (21) on successive sets of three points each. Owing to a known property of polynomial interpolation, the smallest inherent error occurs at the middle point t_n. Therefore, we calculated only the perturbation corresponding to the middle point by the simple formula

$$P(t_n) = [1.5 \ -0.5 \ -0.5 \ 1.5]^T \tilde{R} \tag{38}$$

thus skipping the calculation of $P(t_{n-1})$ and $P(t_{n+1})$.

The following set of three points overlapped on two points with the previous set, and again we applied Eq. (38) to calculate the perturbation for the middle point of the new set, and so forth. In this way, the effect of the inherent error was reduced substantially, although at the cost of an increase in computational effort.

In that manner, we were able to calculate step by step a discrete set of estimates of the perturbation that can be represented by a vector $P = [P_1 \ P_2 \ \ldots \ P_N]^T$.

In our numerical experiments, we adopted for the perturbation P Eq. (32); then we could establish a linear system of equations of condition

$$P = gAE + r \tag{39}$$

where A is an $(N \times 3)$ matrix of the form

$$A = \begin{bmatrix} \sin(\omega_y t_1) & \cos(\omega_y t_1) & \frac{g}{2}\sin(2\omega_y t_1) \\ \vdots & \vdots & \vdots \\ \sin(\omega_y t_N) & \cos(\omega_y t_N) & \frac{g}{2}\sin(2\omega_y t_N) \end{bmatrix} \tag{40}$$

$$E = [E_x, E_z, E_{xz}]^T \tag{41}$$

and r is an $(N \times 1)$ vector of residuals. Then the system (39) is overdetermined, and one can obtain for E a least-squares solution

$$\bar{E} = \frac{1}{g}(A^T A)^{-1} A^T P \tag{42}$$

that minimizes the cost function

$$J = (P - A\bar{E})^T (P - A\bar{E}) \tag{43}$$

Finally, to have an assessment of the accuracy of our results, we may calculate the differences $\delta P_i = P_i - P(t_i)$, where $P(t_i)$ is given by Eq. (32).

A good method for appreciating at a glance the behavior of the differences δP_i is to establish the number of significant figures that are coincident in P_i and $P(t_i)$ by the empirical formula

$$EFF_i = -\log_{10}(|\delta P_i|/10^q \tag{44}$$

where q is the smallest of the exponents of P_i and $P(t_i)$ in their floating point expressions. When $EFF_i > 0$, its integer part gives the number of coincident figures in P_i and $P(t_i)$; if the integer part of $EFF_i < 0$, it indicates that the order of magnitude of P_i and $P(t_i)$ disagree, thus indicating that P_i is a bad estimate.

As an indicator of the accuracy of $\bar{E}_x$, $\bar{E}_z$, and $\bar{E}_{xz}$, we have used the percentage errors

$$e_x = \frac{\bar{E}_x - E_x}{E_x} 100$$

$$e_z = \frac{\bar{E}_z - E_z}{E_z} 100$$

$$e_{xz} = \frac{\bar{E}_{xz} - E_{xz}}{E_{xz}} 100 \tag{45}$$

In our machine runs, our variables were the following:

Input data:

WX = input angular velocity in the case of a gyroscope and input accleration in the case of an accelerometer which, in practice, are simulated with the torque generator T_m
SF = scale factor of the sensor
WN = natural frequency of the sensor, Hz
A = damping factor of the sensor
N = number of points at which the perturbation is determined
T = sampling period, s
Y = initial value of $y(t)$
YP = initial value of $\dot{y}(t)$
WY = angular velocity of the test table about the y axis, deg/s
SI = σ of measurement noise, V
EX = acceleration error along x axis, deg/s/g
EZ = accleration error along z axis, deg/s/g
EXZ = anisoelasticity error factor, deg/s/g^2

Output results:

$P(t)$ = actual value of the perturbation, as given in Eq. (32)
P_i = estimated value of the perturbation
EFF_i = efficiency in the estimation P_i as given in eq. (44)
MQP = mean quadratic value of δP_i
EEX = least-squares estimation of $\bar{E}_x$
EEZ = least-squares estimation of $\bar{E}_z$
EEXZ = least-squares estimation of $\bar{E}_{xz}$
PEX = percentage error of $\bar{E}_x$
PEZ = percentage error of $\bar{E}_z$
PEXZ = percentage error of $\bar{E}_{xz}$

Table 1 Results of numerical experiments

No. of samples	Sampling period h, ms	EFF^a	e_x	e_y	e_{xz}	Assumed variance of measurement errors
250	0.3	6.6	3×10^{-5}	3×10^{-5}	3×10^{-5}	
99	1	4.5	0.003	0.003	0.003	
250	3	2.6	0.275	0.277	0.275	$\sigma=0$
250	6	1.3	4.843	4.892	4.877	
250	1	2.8	−0.041	0.003	0.056	
250	3	2.5	0.275	0.278	0.274	$\sigma=10^{-7}$
250	6	1.3	4.843	4.892	4.878	
250	1	2.0	−0.437	0.003	5.236	
250	3	2.5	0.277	0.282	0.193	$\sigma=10^{-6}$
250	6	1.3	4.840	4.896	4.915	
250	1	1.4	−4.402	0.005	5.237	
250	3	1.7	0.290	0.331	0.193	$\sigma=10^{-5}$
250	6	1.3	4.838	4.870	4.915	
250	3	0.6	0.417	0.815	−7.92	$\sigma=10^{-4}$

[a]Average efficiency in the estimations of $P(t)$ [Eq. (44)] e_x,e_y,e_z: percent errors in the estimations of E_x,E_y,E_{xz} [Eq. (45)].

VI. Numerical Experiments and Conclusions

In our numerical experiments, the inertial sensor was simulated by means of Eq. (34). Then we used our method to estimate the small perturbations and subsequently the coefficients E_x, E_z, and E_{xz}, as described in Secs. II–V. In all cases, our simulator was defined by the following set of parameters:

Desired input angular velocity ω_x: 30 deg/s

Scale factor of sensor $SG=0.1$ V/deg/s

Natural frequency of sensor $\omega_n=30$ Hz

Damping factor of sensor $\xi=0.7$

Initial value $y(0)=0.0$ V

Initial value $\dot{y}(0)=0.0$ V

Angular velocity of the test table $\omega_y=200$ deg/s

$E_x=0.05$ deg/s/g

$E_z=0.05$ deg/s/g

$E_{xz}=0.005$ deg/s/g^2

In each experiment, we took different values of the variance σ of the "measurement errors" applied in Eq. (37) and also different values of the sampling interval of the measurements. With the particular values of the parameters just given, the perturbation to be determined by our method was of the order of 0.001 V/s^2, 3000 times smaller than the main signal. The ω_y angular rate of the table was used to simulate sinusoidal acceleration inputs to the x and z axes of the sensor, by using vertical gravity.

The most significant results of our numerous numerical experiments are summarized in Table 1. From our results from Sec. III concerning upper bounds for the errors, one may expect that for zero or small measurement errors, our method performs best for the smallest sampling period (h), which is clearly shown in the sections corresponding to $\sigma=0$ and $\sigma=10^{-7}$ of Table 1. Note that our minimum period of sampling was imposed by the assumed intrinsic ability of the measurement instruments. Anyway, for small values of σ, even for $h=6$ ms, our results are reasonably accurate.

Table 2 Case of sudden increase of $P(t)$ in a short interval

Sampling time, s	Real values of $P(t)$	EFF
0.095	0.0078946	4.5
0.096	0.0079191	4.5
0.097	0.0079434	4.5
0.098	0.0079676	4.5
0.099	0.0079917	4.5
0.100	0.0080155	−3.7
*0.101	0.8039256	−1.7
0.102	0.8062822	4.5
0.103	0.8086230	4.5
0.104	0.8109479	4.5
0.105	0.8132568	4.5
0.106	0.8155498	4.5
0.107	0.8178266	4.5
0.108	0.8200873	4.5
*0.109	0.8223317	−1.7
0.110	0.0082456	−3.7
0.111	0.0082667	4.5
0.112	0.0082897	4.5
0.113	0.0083115	4.5
0.114	0.0083331	4.5
0.115	0.0083545	4.5

*Ends of the interval.

For larger measurement errors, that is, for $10^{-5}\leq\sigma\leq10^{-4}$, the best results are obtained for an intermediate value of $h=3$ ms.

In Table 2, we show another important feature of our method, which is the ability to detect important changes or irregularities in the unknown perturbations that may appear in very short intervals. In such cases, the standard method of adjusting by a least-squares solution some parameters involved in a predesigned model tend to smooth out those irregularities which nevertheless may affect the final results badly.

We performed an experiment by suddenly increasing the perturbation $P(t)$ by a factor of 10^2 in a short interval of 9 ms and assuming $\sigma=0$.

In all cases, the efficiency of the method was constantly equal to 4.5 decimals, which appear correctly estimated. The bad results that appear at the extremes of the transition interval are evidently due to the inability of the quadratic interpolant to represent a sudden change in P.

We want to emphasize that the method presented here is essentially deterministic. One could approach the same kind of problem from a stochastic standpoint by assuming, for instance, that the unknown perturbing function may be approximated by a convenient Gauss-Markov sequence separated into a function of time, as in our method, plus a purely random component.

This has already been done for similar but more complicated problems by Ingram and Tapley[1] and Tapley and Schutz.[2] This method may eventually draw more information from the measured data at the cost of a larger computational effort. The philosophy underlying our deterministic method is as follows.

Accuracy in the results of a stochastic method, be it a least-squares process or a filtering technique, used for parameter identifications may be badly affected if the assumed models for both the dynamic system and the behavior of the error are inadequate. Our deterministic method is based only on the simple assumptions that the solutions of the differential equations can be expressed piecewise in short intervals by a Taylor convergent expansion and similarly that the unknown perturbations can also be approximated piecewise by polynomials or eventually by a combination of other elementary functions. The deterministic method presented here may be used as a first approach to a complicated problem, and its results may allow an adquate model to be built up for later application of a more refined stochastic technique.

References

[1]Ingram, D. S. and Tapley, B. D., "Lunar Orbit Determination in the Presence of Unmodeled Accelerations," *Celestial Mechanics*, Vol. 9, 1974, pp. 191-211.

[2]Tapley, B. D. and Schutz, B. E., "Estimation of Unmodeled Forces on a Lunar Satellite," *Celestial Mechanics*, Vol. 12, 1975, pp. 440-442.

[3]Zadunaisky, P. E., "A Method for the Estimation of Small Perturbations," *Proceedings of the Conference on the Motion of Planets and Natural and Artificial Satellites*, edited by S. Ferraz-Mello and P. Nacozi, Universidade de São Paulo, Brazil, 1983, pp. 91-102.

[4]Zadunaisky, P. E., "On the Estimation of Small Perturbations in Ordinary Differential Equations," International Workshop on "Numerical Treatment of Inverse Problems in Differential and Integral Equations," edited by P. Deuflhard and E. Hairer, Birkhauser, Universität Heidelberg, FRG, 1983, pp. 62-72.